# MEA714: Review of cloud/meso–scale governing equations

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**The purpose of this document** is to show the details necessary to derive some key results that appear in Houze's *Cloud Dynamics* textbook. You have been assigned to read Houze's Chapter 7. We also recommend that you review Houze's Chapter 2 (download it from the from course website), which is a review of many of the basic equations that are used in later chapters of the text.

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**The primary variables of interest are:**

$u$, $v$, $w$, $p$, $\theta$, $\alpha$ ($1/\rho$), $q_v$, and $q_n$ (where n = 1, # of hydrometeor species).

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**Vector equation for motion:**

$$\frac{D\mathbf{u}}{Dt} = -\alpha\boldsymbol{\nabla}p - \mathbf{g}^* + \mathbf{F_r} - 2\boldsymbol{\Omega}\times\mathbf{u} - \boldsymbol{\Omega}\times\boldsymbol{\Omega}\times\mathbf{r} = -\alpha\boldsymbol{\nabla}p - \mathbf{g} + \mathbf{F_r} - 2\boldsymbol{\Omega}\times\mathbf{u}. \quad (1)$$

This is equivalent to Houze's (2.1), wherein vectors are denoted by bold type; $\mathbf{u}$ is the wind vector, with components $u$, $v$, and $w$; $\mathbf{g}^*$ is Newton's gravitational acceleration, $\mathbf{g} = \mathbf{g}^* + \boldsymbol{\Omega}\times\boldsymbol{\Omega}\times\mathbf{r}$ is the apparent gravitational acceleration (including Newton's gravity and the centrifugal acceleration aassociated with planetary rotation); $\boldsymbol{\Omega}$ is the planetary rotation vector, **Fr** is the frictional acceleration vector; and $D/Dt$ is the material (parcel–following) derivative operator, $D/Dt = \partial/\partial t + \mathbf{u}\cdot\boldsymbol{\nabla}$. The components of this equation are:

$$\frac{Du}{Dt} = -\alpha\frac{\partial p}{\partial x} - 2w\Omega\cos\phi + 2v\Omega\sin\phi + F_{r\ x}; \quad (2)$$

$$\frac{Dv}{Dt} = -\alpha\frac{\partial p}{\partial y} - 2u\Omega\sin\phi + F_{r\ y}; \quad (3)$$

$$\frac{Dw}{Dt} = -\alpha\frac{\partial p}{\partial z} - g + 2u\Omega\cos\phi + F_{r\ z}; \quad (4)$$

wherein $\phi$ is latitude. Scale analysis shows that for *mid–latitude* cloud–scale and mesoscale analysis of moist convection, we may simplify to:

$$\frac{Du}{Dt} = -\alpha\frac{\partial p}{\partial x} + fv; \quad (5)$$

$$\frac{Dv}{Dt} = -\alpha\frac{\partial p}{\partial y} - fu; \quad (6)$$

$$\frac{Dw}{Dt} = -\alpha\frac{\partial p}{\partial z} - g; \quad (7)$$

wherein $f = 2\Omega\sin\phi$ is the Coriolis parameter.

We wish to apply our equations to convective clouds, and an obvious characteristic of cloudy air is that hydrometeors (liquid and solid water) are present. The mass of hydrometeors adds to the weight of an air parcel. To include this effect, consider (7) in terms of forces per unit volume (rather than in terms of accelerations):

$$\rho\,\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} - \rho g. \quad (8)$$

If $q_H$ is the mixing ratio of all hydrometeors in the air, the mass per unit volume of all the hydrometeors is $\rho q_H$. Thus the force associated with the weight of the hydrometeors is included via:

$$\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} - \rho g - \rho q_H g; \tag{9}$$

or, in terms of accelerations:

$$\frac{Dw}{Dt} = -\alpha \frac{\partial p}{\partial z} - g\left(1 + q_H\right). \tag{10}$$

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**Conservation of mass:**

$$\frac{D\rho}{Dt} = -\rho \boldsymbol{\nabla} \cdot \mathbf{u}; \tag{11}$$

which is equivalent to Houze's (2.20). Or, alternatively:

$$\frac{D\alpha}{Dt} = \alpha \boldsymbol{\nabla} \cdot \mathbf{u}. \tag{12}$$

Consider splitting $\alpha$ into mean and perturbation parts ($\alpha = \alpha_o + \alpha'$); for most reasonable (i.e. non–nuclear/volcanic) convective clouds, $\alpha' \ll \alpha_o$. In other words, $\alpha_o + \alpha' \approx \alpha_o$. Therefore:

$$\mathbf{u} \cdot \boldsymbol{\nabla} \alpha_o = \alpha_o \boldsymbol{\nabla} \cdot \mathbf{u}; \tag{13}$$

or, alternatively:

$$\boldsymbol{\nabla} \cdot \left(\rho_o \mathbf{u}\right) = 0. \tag{14}$$

This is the *anelastic (sound–proof)* continuity equation, which is equivalent to Houze's (2.54). Because the mean $\rho_o$ and $\alpha_o$ are only functions of height, if desired (13) and (14) can also be written:

$$w\frac{d\alpha_o}{dz} - \alpha_o \boldsymbol{\nabla} \cdot \mathbf{u} = w\frac{d\rho_o}{dz} + \rho_o\left(\boldsymbol{\nabla} \cdot \mathbf{u}\right) = 0. \tag{15}$$

For some applications, (14) can be further simplified via the *Boussinesq* approximation, in which $\rho_o$ is treated as a constant. In this case, the continuity equation expresses three–dimensional incompressibility, $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$, which is equivalent to Houze's (2.55). Treating density as invariant in the vertical (the Boussinesq approximation) can be acceptable when studying shallow motions. However, it is a poor assumption when studying deep motions (e.g., cumulonimbi).

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**Thermodynamic equation:**

$$\frac{D\theta}{Dt} = S_\theta, \tag{16}$$

where $S_\theta$ is a source or sink term (e.g., diabatic heating). This is equivalent to Houze's (2.9).

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**Conservation of water substance:**

$$\frac{Dq_v}{Dt} = S_{q_v}; \tag{17}$$

$$\frac{Dq_n}{Dt} = S_{q_n}, \tag{18}$$

for n = 1, # of hydrometeor species. Again, $S_{q_*}$ is a source or sink term related to conversions among species. This is equivalent to Houze's (2.21).

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**Equation of state:**

$$p\alpha = R'T, \tag{19}$$

wherein $R'$ is the gas constant for the moist air mixture, and $\alpha$ is understood to represent the parcel's specific volume considering all gaseous species, including water vapor. For simplicity, we commonly use the virtual temperature:

$$p\alpha = R_d T_v, \tag{20}$$

which is equivalent to Houze's (2.4), and wherein:

$$T_v = T\left(\frac{1+q_v/\epsilon}{1+q_v}\right) \approx \theta\left(\frac{p}{100000\ \text{Pa}}\right)^{R/c_p}(1+0.61q_v). \tag{21}$$

Splitting (20) into mean and perturbation parts:

$$p_o\left(1+p'/p_o\right)\ \alpha_o\left(1+\alpha'/\alpha_o\right) = R_d\, T_{v_o}\left(1+T_v'/T_{v_o}\right). \tag{22}$$

Removing the base state (i.e. dividing by $p_o\alpha_o = R_d T_{v_o}$) and taking the logarithm of (22) yields:

$$\ln\left(1+p'/p_o\right) + \ln\left(1+\alpha'/\alpha_o\right) = \ln\left(1+T_v'/T_{v_o}\right). \tag{23}$$

Provided that the perturbations are reasonably small, a Taylor series approximation [$\ln(1+x) \approx x - x^2/2 + x^3/3 - x^4/4 + ...$] can be used to rewrite (23) to a very good approximation as:

$$\frac{p'}{p_o} + \frac{\alpha'}{\alpha_o} = \frac{T_v'}{T_{v_o}}, \tag{24}$$

which is equivalent to Houze's (2.74). This form often appears in cloud–scale analyses. The equation of state is also occasionally written using the virtual potential temeprature,

$$\theta_v = T_v\left(\frac{100000\ \text{Pa}}{p}\right)^{R/c_p} \approx \theta(1+0.61q_v). \tag{25}$$

Substituting and following through the preceding analysis then yields:

$$\left(1-\frac{R}{c_p}\right)\frac{p'}{p_o} + \frac{\alpha'}{\alpha_o} = \frac{\theta_v'}{\theta_{v_o}}. \tag{26}$$

This form is also used in cloud–scale analyses, e.g., the buoyancy term in Houze's (2.52).

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**Anelastic equations:**

The above represent a closed set (with the exception of the $S_*$ terms, which are specified from thermodynamic/microphysical equations) in the variables $u$, $v$, $w$, $p$, $\theta$, $\alpha$, $q_v$, and $q_n$. Because acoustic waves are generally thought to be meteorologically irrelevant, it is often preferable to use (13) or (14). For consistency with the anelastic continuity equation, the assumption that $\alpha_o + \alpha' \approx \alpha_o$ should also be carried through in in the equations of motion. For (5) and (6) this is simply:

$$\frac{Du}{Dt} = -\left(\alpha_o+\alpha'\right)\frac{\partial}{\partial x}\left(p_o+p'\right) + fv \approx -\alpha_o\frac{\partial p'}{\partial x} + fv; \tag{27}$$

$$\frac{Dv}{Dt} = -\left(\alpha_o+\alpha'\right)\frac{\partial}{\partial y}\left(p_o+p'\right) - fu \approx -\alpha_o\frac{\partial p'}{\partial y} - fu. \tag{28}$$

Note again that the horizontal gradients of all mean atmospheric variables (e.g. $p_o$) are zero.

More care is required in the case of $w$, so that small but potentially important buoyant accelerations are retained. Rewriting (10):

$$\frac{Dw}{Dt} = -\left(\alpha_o + \alpha'\right)\frac{\partial}{\partial z}\left(p_o + p'\right) - g\left(1 + q_{_H}\right). \tag{29}$$

The base state is hydrostatic and free of hydrometeors ($\alpha_o\ \partial p_o/\partial z = -g$); removing this hydrostatic mean state yields:

$$\begin{aligned} \frac{Dw}{Dt} &= -\left(\alpha_o + \alpha'\right)\frac{\partial p'}{\partial z} - \alpha'\frac{\partial p_o}{\partial z} - gq_{_H} && (30)\\ &= -\left(\alpha_o + \alpha'\right)\frac{\partial p'}{\partial z} + \alpha'\frac{g}{\alpha_o} - gq_{_H} && (31)\\ &= -\left(\alpha_o + \alpha'\right)\frac{\partial p'}{\partial z} + g\left(\frac{\alpha'}{\alpha_o} - q_{_H}\right) && (32)\\ &\approx -\alpha_o\frac{\partial p'}{\partial z} + g\left(\frac{\alpha'}{\alpha_o} - q_{_H}\right). && (33) \end{aligned}$$

This formulation at last explicitly reveals the concept of buoyancy in the vertical accelerations. Using (24) or (26), buoyancy can be defined as:

$$B \equiv g\left(\frac{\alpha'}{\alpha_o} - q_{_H}\right) = g\left(\frac{T_v'}{T_{v_o}} - \frac{p'}{p_o} - q_{_H}\right) = g\left[\frac{\theta_v'}{\theta_{v_o}} + \left(\frac{R}{c_p} - 1\right)\frac{p'}{p_o} - q_{_H}\right], \tag{34}$$

so that (33) is more compactly written:

$$\frac{Dw}{Dt} = -\alpha_o\frac{\partial p'}{\partial z} + B. \tag{35}$$

Collecting (27), (28), and (35), the vector equation of motion that is compatible with the anelastic continuity equation is:

$$\frac{D\mathbf{u}}{Dt} = -\alpha_o\boldsymbol{\nabla}p' - f\mathbf{k}\times\mathbf{u} + B\mathbf{k}. \tag{36}$$

This is equivalent to Houze's (7.2), except that Houze has ignored the Coriolis term. If we make the Boussinesq (shallow motion) approximation, (36) continues to apply to the horizontal wind components $u$ and $v$. However, the Boussinesq continuity equation, $\boldsymbol{\nabla}\cdot\mathbf{u} = 0$, constrains the relationship among $u$, $v$, and $w$, so that we can only predict two of the three. Therefore, in the Boussinesq system, $w$ is not predicted from (36). Instead, it is determined diagnostically from $u$ and $v$ using the continuity equation.

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**Diagnostic pressure equation:**
The anelastic set has the benefit that sound waves are excluded. But, it is no longer possible to predict the pressure field because the total density is never predicted when using (13) or (14). However, the pressure field must be obtained in order to evaluate the pressure gradient acceleration in (36). For this reason, a diagnostic pressure equation is needed. Moreover, the diagnostic pressure equation proves quite useful for cloud–scale analysis. Combining $\mathbf{u}$(14) with $\rho$(36) converts the vector equation for motion to *flux form*:

$$\frac{\partial}{\partial t}\left(\rho_o\mathbf{u}\right) + \boldsymbol{\nabla}\cdot\left(\rho_o\mathbf{u}\mathbf{u}\right) = -\boldsymbol{\nabla}p' - \rho_o f\mathbf{k}\times\mathbf{u} + \rho_o B\mathbf{k}. \tag{37}$$

The components of this vector equation are:

$$\frac{\partial}{\partial t}\left(\rho_o u\right) + \boldsymbol{\nabla}\cdot\left(\rho_o u\mathbf{u}\right) + \frac{\partial p'}{\partial x} - \rho_o f v = 0; \tag{38}$$

$$\frac{\partial}{\partial t}(\rho_o v) + \boldsymbol{\nabla}\cdot(\rho_o v\mathbf{u}) + \frac{\partial p'}{\partial y} + \rho_o f u = 0; \tag{39}$$

$$\frac{\partial}{\partial t}(\rho_o w) + \boldsymbol{\nabla}\cdot(\rho_o w\mathbf{u}) + \frac{\partial p'}{\partial z} - \rho_o B = 0. \tag{40}$$

The diagnostic pressure equation is derived by taking $\boldsymbol{\nabla}\cdot$ (37), in other words, by combining $\partial/\partial x$(38), $\partial/\partial y$(39), and $\partial/\partial z$(40). The resultant component equations are:

$$\frac{\partial}{\partial t}\left[\frac{\partial}{\partial x}(\rho_o u)\right] + \frac{\partial}{\partial x}\left[u\underbrace{\boldsymbol{\nabla}\cdot(\rho_o\mathbf{u})}_{=0} + \rho_o\mathbf{u}\cdot\boldsymbol{\nabla}u\right] + \frac{\partial^2 p'}{\partial x^2} - \rho_o\frac{\partial}{\partial x}(fv) = 0; \tag{41}$$

$$\frac{\partial}{\partial t}\left[\frac{\partial}{\partial y}(\rho_o v)\right] + \frac{\partial}{\partial y}\left[v\underbrace{\boldsymbol{\nabla}\cdot(\rho_o\mathbf{u})}_{=0} + \rho_o\mathbf{u}\cdot\boldsymbol{\nabla}v\right] + \frac{\partial^2 p'}{\partial y^2} + \rho_o\frac{\partial}{\partial y}(fu) = 0; \tag{42}$$

$$\frac{\partial}{\partial t}\left[\frac{\partial}{\partial z}(\rho_o w)\right] + \frac{\partial}{\partial z}\left[w\underbrace{\boldsymbol{\nabla}\cdot(\rho_o\mathbf{u})}_{=0} + \rho_o\mathbf{u}\cdot\boldsymbol{\nabla}w\right] + \frac{\partial^2 p'}{\partial z^2} - \frac{\partial}{\partial z}(\rho_o B) = 0; \tag{43}$$

or, in vector form:

$$\frac{\partial}{\partial t}\left[\underbrace{\boldsymbol{\nabla}\cdot(\rho_o\mathbf{u})}_{=0}\right] + \boldsymbol{\nabla}\cdot\left[\rho_o(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}\right] + \nabla^2 p' - \frac{\partial}{\partial z}(\rho_o B) - \rho_o\boldsymbol{\nabla}\cdot\left(f\hat{k}\times\mathbf{u}\right) = 0. \tag{44}$$

This is equivalent to Houze's (7.3), except that again Houze has neglected the Coriolis term. Regrouping yields the traditional form of the diagnostic pressure equation:

$$\nabla^2 p' = -\boldsymbol{\nabla}\cdot\left[\rho_o(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}\right] + \frac{\partial}{\partial z}(\rho_o B) + \rho_o\boldsymbol{\nabla}\cdot\left(f\hat{k}\times\mathbf{u}\right). \tag{45}$$

At any time this can be used to infer the perturbation pressure field from other known variables.

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**Interpretation:**

Unfortunately, (45) is not easy to interpret straightaway. For this reason, the perturbation pressure field is often broken into three parts: a part that is in geostrophic balance with the wind field, a part that is due to local buoyancy, and a a part that is due to dynamic effects (i.e. $p' = p'_G + p'_B + p'_D$, and therefore $\nabla^2 p' = \nabla^2 p'_G + \nabla^2 p'_B + \nabla^2 p'_D$). Separating (45) into its geostrophic, buoyant, and dynamic parts yields:

$$\nabla^2 p'_G = \rho_o\boldsymbol{\nabla}\cdot\left(f\hat{k}\times\mathbf{u}\right); \tag{46}$$

$$\nabla^2 p'_B = \frac{\partial}{\partial z}(\rho_o B), \tag{47}$$

which is equivalent to Houze's (7.5); and,

$$\nabla^2 p'_D = -\boldsymbol{\nabla}\cdot\left[\rho_o(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}\right], \tag{48}$$

which is equivalent to Houze's (7.6).

· The geostrophic part in (46) is treated at length in dynamic and synoptic meteorology courses.

· The meaning of (47) is that the pressure is minimized where buoyancy increases with height, and maximized where buoyancy decreases with height.

In order to gain insight from (48), however, some additional work is needed. Applying the vector identity $\boldsymbol{\nabla}\cdot(\psi\mathbf{a}) = \mathbf{a}\cdot\boldsymbol{\nabla}\psi + \psi\left(\boldsymbol{\nabla}\cdot\mathbf{a}\right)$ to (48) yields:

$$-\nabla^2 p'_D = \left[\left(\mathbf{u}\cdot\boldsymbol{\nabla}\right)\mathbf{u}\right]\cdot\boldsymbol{\nabla}\rho_o + \rho_o\boldsymbol{\nabla}\cdot\left[\left(\mathbf{u}\cdot\boldsymbol{\nabla}\right)\mathbf{u}\right], \tag{49}$$

or, as $\rho_o = \rho_o\left(z\right)$,

$$-\nabla^2 p'_D = \left[\left(\mathbf{u}\cdot\boldsymbol{\nabla}\right)w\right]\frac{\partial\rho_o}{\partial z} + \rho_o\boldsymbol{\nabla}\cdot\left[\left(\mathbf{u}\cdot\boldsymbol{\nabla}\right)\mathbf{u}\right]. \tag{50}$$

It proves useful to expand all of the terms in (50), so that they can be regrouped in a more efficacious way:

$$\begin{aligned}
-\nabla^2 p'_D \;=\;& \frac{\partial\rho_o}{\partial z}\left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)\\
&+\rho_o\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + u\frac{\partial^2 u}{\partial x^2} + \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} + v\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial w}{\partial x}\frac{\partial u}{\partial z} + w\frac{\partial^2 u}{\partial x\partial z}\right.\\
&+\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial x\partial y} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial y} + v\frac{\partial^2 v}{\partial y^2} + \frac{\partial w}{\partial y}\frac{\partial v}{\partial z} + w\frac{\partial^2 v}{\partial y\partial z}\\
&\left.+\frac{\partial u}{\partial z}\frac{\partial w}{\partial x} + u\frac{\partial^2 w}{\partial x\partial z} + \frac{\partial v}{\partial z}\frac{\partial w}{\partial y} + v\frac{\partial^2 w}{\partial z\partial y} + \frac{\partial w}{\partial z}\frac{\partial w}{\partial z} + w\frac{\partial^2 w}{\partial z^2}\right),
\end{aligned} \tag{51}$$

or:

$$\begin{aligned}
-\nabla^2 p'_D \;=\;& \frac{\partial\rho_o}{\partial z}\left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)\\
&+\rho_o\left[u\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + v\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + w\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right.\\
&\left.+\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 + 2\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial w}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial w}{\partial y}\right)\right],
\end{aligned} \tag{52}$$

or:

$$\begin{aligned}
-\nabla^2 p'_D \;=\;& \frac{\partial\rho_o}{\partial z}\left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) + \rho_o\left[\mathbf{u}\cdot\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{u}\right)\right]\\
&+\rho_o\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] + 2\rho_o\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial w}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial w}{\partial y}\right).
\end{aligned} \tag{53}$$

The second term on the r.h.s. of (53) can now be re–written using (15):

$$\rho_o\left[\mathbf{u}\cdot\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{u}\right)\right] = -\rho_o\left[\mathbf{u}\cdot\boldsymbol{\nabla}\left(\frac{w}{\rho_o}\frac{\partial\rho_o}{\partial z}\right)\right] \tag{54}$$

$$= -u\frac{\partial\rho_o}{\partial z}\frac{\partial w}{\partial x} - v\frac{\partial\rho_o}{\partial z}\frac{\partial w}{\partial y} - \rho_o w^2\frac{\partial^2}{\partial z^2}\left(\ln\rho_o\right) - w\frac{\partial\rho_o}{\partial z}\frac{\partial w}{\partial z}. \tag{55}$$

Substituting (55) into (53) and cancelling corresponding terms then leaves:

$$\begin{aligned}
\nabla^2 p'_D \;=\;& -\rho_o\underbrace{\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 - w^2\frac{\partial^2}{\partial z^2}\left(\ln\rho_o\right)\right]}_{\text{fluid extension terms}}\\
&-2\rho_o\underbrace{\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial w}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial w}{\partial y}\right)}_{\text{fluid shear terms}}.
\end{aligned} \tag{56}$$

This is the form of the diagnostic equation for dynamic pressure pertubations that is most often given in the literature and in educational textbooks. The meaning of (56) is that the pressure is minimized in regions of vortical flow (i.e. the shear term is $> 0$), and is maximized in regions of convergence (i.e. the extension term is $< 0$) or deformation (i.e. the shear term is $< 0$).

For some analyses, the diagnostic pressure equation is divided into linear and non-linear parts. Dividing the components of the velocity vector into mean and perturbation parts ($u = u_o(z) + u',\ v = v_o(z) + v',\ w = w'$), substituting into (56), and then removing all non–linear terms (i.e. "products of primes") yields:

$$\nabla^2 p'_{D\ lin} = -2\rho_o \left( \frac{du_o}{dz}\frac{\partial w}{\partial x} + \frac{dv_o}{dz}\frac{\partial w}{\partial y} \right). \tag{57}$$

The other remaining terms are therefore the non-linear part:

$$\begin{aligned} \nabla^2 p'_{D\ non} \quad = \quad & -\rho_o \left[ \left(\frac{\partial u'}{\partial x}\right)^2 + \left(\frac{\partial v'}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 - w^2 \frac{\partial^2}{\partial z^2}(\ln \rho_o) \right] \\ & -2\rho_o \left( \frac{\partial v'}{\partial x}\frac{\partial u'}{\partial y} + \frac{\partial u'}{\partial z}\frac{\partial w}{\partial x} + \frac{\partial v'}{\partial z}\frac{\partial w}{\partial y} \right). \end{aligned} \tag{58}$$

This can be a useful separation because $p'_{D\ lin}$ in (57) can be thought of as the pressure perturbation that is due to the presence of an updraft in the base state environmental shear.

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**Vorticity:**
The prognostic equation for vertical vorticity ($\zeta = \mathbf{k} \cdot (\boldsymbol{\nabla} \times \mathbf{u})$) is obtained by taking $\mathbf{k}\cdot$ [$\boldsymbol{\nabla}\times$ (36)], or equivalently by combining $\partial/\partial x$(28) $-\partial/\partial y$(27):

$$\frac{D}{Dt}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = -\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \left(\frac{\partial w}{\partial y}\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\frac{\partial v}{\partial z}\right) - f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - v\frac{df}{dy}, \tag{59}$$

or:

$$\frac{D\zeta}{Dt} = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \left(\frac{\partial w}{\partial y}\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\frac{\partial v}{\partial z}\right) \underbrace{-v\frac{df}{dy}}_{\text{(often neglected)}}. \tag{60}$$

This equation is equivalent to Houze's (2.59), if we make the Boussinesq approximation and treat $f$ as a constant (as Houze does). The vorticity equation should be familiar from dynamic and synoptic meteorology. As indicated, the effect of $f$'s change with latitude is usually neglected as small in cloud and mesoscale studies.

The prognostic equation for horizontal vorticity about the $y$–axis ($\xi = \mathbf{j} \cdot (\boldsymbol{\nabla} \times \mathbf{u})$) is often of interest in quasi–2D squall line studies (in which $x$ is usually taken to be the across–line dimension, and $y$ is taken to be the along–line dimension). It is obtained by taking **j**$\cdot$ [$\boldsymbol{\nabla}\times$ (36)], or equivalently by combining $\partial/\partial z$(27) $-\partial/\partial x$(29):

$$\frac{D}{Dt}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = -\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) + \left(\frac{\partial v}{\partial x}\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\frac{\partial u}{\partial y}\right) - \frac{d\alpha_o}{dz}\frac{\partial p'}{\partial x} + f\frac{dv}{dz} - \frac{\partial B}{\partial x}, \tag{61}$$

or:

$$\frac{D\xi}{Dt} = \overbrace{-\frac{\partial B}{\partial x}}^{(1)} \underbrace{\overbrace{+f\frac{\partial v}{\partial z}}^{(2)} \overbrace{+\left(\frac{\partial v}{\partial x}\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\frac{\partial u}{\partial y}\right)}^{(3)} \overbrace{-\frac{d\alpha_o}{dz}\frac{\partial p'}{\partial x}}^{(4)} \overbrace{-\xi\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right)}^{(5)}}_{\text{often neglected, e.g., Houze's (2.61)}}, \tag{62}$$

This is equivalent to Houze's (2.58), if we make the Boussinesq approximation (as Houze does). Term 1 on the r.h.s. shows that horizontal gradients in buoyancy baroclinically generate horizontal vorticity. This term is often thought to predominate in cloud and mesoscale studies and, as indicated, the other terms on the r.h.s. have frequently been ignored. Term 2 is the twisting of planetary vorticity into the horizontal; it is often neglected either because it is much smaller than the twisting of the relative vertical vorticity in term 3, or in squall line studies because the vertical shear of the along–line wind (i.e. $\partial v/\partial z$) is small or is set to zero. Term 3 is the twisting of off–axis vorticity into the $y$–dimension; it has often been ignored because in 2D squall line studies there are no $y$–variations (i.e. $\partial/\partial y = 0$). Term 4 is the solenoidal generation of horizontal vorticity owing to the vertical gradient in $\alpha_o$; it is often assumed to be small. Term 5 is the convergence of horizontal vorticity; this term could be significant in some cases (picture a gust front), but it has been neglected by many 2D squall line investigators who assume that the flow is incompressible (if $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$ and $\partial/\partial y = 0$ then $\partial u/\partial x + \partial w/\partial z = 0$).