Therefore, the entire RHS in (3.11) is:

$$-c_{pd}\bar{\theta}_v\frac{\partial\pi'}{\partial z}+g\frac{\theta_v'}{\bar{\theta}_v} \;=\; -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial z}-\frac{g}{\bar{c_s}^2}\frac{p'}{\bar{\rho}}+g\frac{\theta_v'}{\bar{\theta}_v}. \tag{3.21}$$

Thus, it is seen that the right hand sides of (3.11) and (3.16) are at the same level of approximation.

3.4 Converting the continuity equation into a pressure tendency equation

3.4.1 Derivation

As mentioned before, we do not need prognostic equations for both density and pressure, owing to the ideal gas law. Because perturbation pressure appears in important terms in the model, and because we've already replaced the buoyancy term's perturbation density with θ_v', it is convenient to replace our prognostic equation for density [the continuity equation, (3.4)] with an equation for the nondimensional pressure tendency.

Using the ideal gas law we can manipulate (3.6) into:

$$\pi^{\frac{c_{vd}}{R_d}} \;=\; \frac{\rho R_d\theta_v}{p_0}. \tag{3.22}$$

Take the substantial derivative of this expression and manipulate it into this form:

$$\begin{aligned}\frac{c_{vd}}{R_d}\pi^{\frac{c_{vd}}{R_d}-1}\frac{d\pi}{dt} &= \frac{\rho R_d\theta_v}{p_0}\left[\frac{1}{\rho}\frac{d\rho}{dt}+\frac{1}{\theta_v}\frac{d\theta_v}{dt}\right]\\ &= \pi^{\frac{c_{vd}}{R_d}}\left[\frac{1}{\rho}\frac{d\rho}{dt}+\frac{1}{\theta_v}\frac{d\theta_v}{dt}\right]\end{aligned} \tag{3.23}$$

where the latter expression was obtained through usage of (3.22). Divide through by $\pi^{\frac{c_{vd}}{R_d}}$ and rearranging a little yields this expression:

$$\frac{d\pi}{dt} \;=\; \frac{R_d\pi}{c_{vd}}\left[\frac{1}{\rho}\frac{d\rho}{dt}+\frac{1}{\theta_v}\frac{d\theta_v}{dt}\right] \tag{3.24}$$

At this point, use the continuity equation (3.4) to replace the density derivative on the RHS. Recalling that the adiabatic speed of sound is $c_s^2=\frac{c_{pd}}{c_{vd}}R_d\theta_v\pi$, we see that

$$\frac{R_d\pi}{c_{vd}} = \frac{c_s^2}{c_{pd}\theta_v}. \tag{3.25}$$

Using (3.25), we have:

$$\frac{d\pi}{dt} = -\frac{R_d\pi}{c_{vd}}\nabla\cdot\vec{V} + \frac{c_s^2}{c_{pd}\theta_v}\frac{d\theta_v}{dt}. \tag{3.26}$$

Now subject (3.26) to a perturbation analysis, recalling that the mean state is a function of height alone, except we're only going to (explicitly) expand π for now. The LHS of (3.26) becomes:

$$\begin{aligned}\frac{d\pi}{dt} &= \frac{\partial\pi}{\partial t} + u\frac{\partial\pi}{\partial x} + w\frac{\partial\pi}{\partial z}\\ &= \frac{\partial\pi'}{\partial t} + u\frac{\partial\pi'}{\partial x} + w\frac{\partial\pi'}{\partial z} + w\frac{d\bar{\pi}}{dz}\\ &= \frac{\partial\pi'}{\partial t} + \vec{V}\cdot\nabla\pi' + w\frac{d\bar{\pi}}{dz}.\end{aligned} \tag{3.27}$$

The RHS of (3.26) becomes:

$$-\frac{R_d\bar{\pi}}{c_{vd}}\nabla\cdot\vec{V} - \frac{R_d\pi'}{c_{vd}}\nabla\cdot\vec{V} + \frac{c_s^2}{c_{pd}\theta_v}\frac{d\theta_v}{dt}. \tag{3.28}$$

One term from each side may be combined in the following fashion:

$$\left[-\frac{R_d\bar{\pi}}{c_{vd}}\nabla\cdot\vec{V} - w\frac{d\bar{\pi}}{dz}\right] \Rightarrow -\frac{\bar{c_s}^2}{\bar{\rho}c_{pd}\bar{\theta}_v^2}\left[\nabla\cdot\bar{\rho}\bar{\theta}_v\vec{V}\right], \tag{3.29}$$

which accomplishes an important simplification. Note the mean state sound speed defined in (3.17) has been employed.

It is easier to show (3.29) is true by going backwards. First, apply the vector chain rule to the RHS of (3.29):

$$\frac{\bar{c_s}^2}{\bar{\rho}c_{pd}\bar{\theta}_v^2}\left[\bar{\rho}\bar{\theta}_v\nabla\cdot\vec{V} + \vec{V}\cdot\nabla\bar{\rho}\bar{\theta}_v\right] \tag{3.30}$$

Recognizing the mean fields vary only vertically, we obtain:

$$\frac{\bar{c_s}^2}{c_{pd}\bar{\theta}_v}\nabla\cdot\vec{V} + \frac{\bar{c_s}^2}{\bar{\rho}c_{pd}\bar{\theta}_v^2}w\frac{d\bar{\rho}\bar{\theta}_v}{dz} \tag{3.31}$$

Use (3.25) and split the vertical derivative in the second term to yield:

$$\underbrace{\frac{R_d\bar{\pi}}{c_{vd}}\nabla\cdot\vec{V}}_{A} + \frac{R_d\bar{\pi}}{c_{vd}\bar{\rho}\bar{\theta}_v}w\left[\bar{\rho}\frac{d\bar{\theta}_v}{dz} + \bar{\theta}_v\frac{d\bar{\rho}}{dz}\right] \tag{3.32}$$

which can be manipulated into:

$$A + \frac{R_d \bar{\pi}}{c_{vd}} w \left[\frac{d \ln \bar{\theta}_v}{dz} + \frac{d \ln \bar{\rho}}{dz} \right]. \tag{3.33}$$

Now, the term in square brackets has been seen before, in (3.20). Use (3.20) to replace the density derivative in (3.33) above. We should find that the two terms with the vertical temperature derivatives exactly cancel, leaving us with:

$$A - \frac{R_d \bar{\pi}}{c_{vd}} w \frac{g}{\bar{c_s}^2} \tag{3.34}$$

Take this expression, the hydrostatic equation (3.10) and the base state sound speed (3.17), and we finally end up with

$$-\frac{\bar{c_s}^2}{\bar{\rho} c_{pd} \bar{\theta}_v^2} \left[\nabla \cdot \bar{\rho} \bar{\theta}_v \vec{V} \right] = -\frac{R_d \bar{\pi}}{c_{vd}} \nabla \cdot \vec{V} - w \frac{d\bar{\pi}}{dz}, \tag{3.35}$$

which concludes the proof.

3.4.2 Interpretation

This is the pressure tendency equation we have derived:

$$\frac{\partial \pi'}{\partial t} = \underbrace{-\vec{V} \cdot \nabla \pi'}_{[1]} \underbrace{- \frac{\bar{c_s}^2}{\bar{\rho} c_{pd} \bar{\theta}_v^2} \left[\nabla \cdot \bar{\rho} \bar{\theta}_v \vec{V} \right]}_{[2]} - \frac{R_d \pi'}{c_{vd}} \nabla \cdot \vec{V} + \underbrace{\frac{c_s^2}{c_{pd} \theta_v^2} \frac{d\theta_v}{dt}}_{[3]}. \tag{3.36}$$

The left side, of course, represents the local perturbation pressure rate of change at a single location. Three terms on the RHS have been highlighted. These terms influence the local pressure tendency through:

Term 1 *Advection of pressure.* "The future pressure here depends on the present pressure there". The local pressure rises or falls depending on what the wind is carrying with it. Advection travels with the flow velocity, with a time scale given by $\frac{L}{V}$, where L is some measure of horizontal length (such as the grid spacing) and V is the flow speed. Since $V \approx 10$ m s^{-1} or so, pressure advection is not rapid.

Term 2 *Acoustic adjustment.* "The future pressure here depends on the present wind field here (the dynamics)". The local pressure rises or falls depending (primarily) on the local mass convergence. This signal travels at the sound wave speed, so its time scale is $\frac{L}{c_s}$ (i.e., quite short).

Term 3 *Diabatic heating.* "The future pressure here depends on the thermodynamics."

Rewrite (3.36), incorporating all terms on the RHS except term [1] into catch-all proxy f_π, and we have the equation in the form we plan to use in the model:

$$\frac{\partial \pi'}{\partial t} + \frac{\bar{c_s}^2}{\bar{\rho} c_{pd} \bar{\theta}_v^2} \left[\nabla \cdot \bar{\rho} \bar{\theta}_v \vec{V} \right] = f_\pi. \tag{3.37}$$

In their 3D model, Klemp and Wilhelmson (1978, *J. Atmos. Sci.*, p. 1070) argued that the terms embodied in f_π appear to have little effect on important physical processes operating in convection, and thus set the term to zero. (They remarked that this would be formally justified, but the paper they referred to never appeared — at least to my knowledge.) The chief consequence of neglecting f_π appears to be that unique values of pressure are no longer obtained; instead, the field is predicted only to within an unknown constant. Since we have successfully avoided having to use raw values of π', this is not a concern. We will follow Klemp and Wilhelmson and set f_π to zero.

3.5 The fully compressible model equations

These are the equations we've developed thusfar (after performing a compatible perturbation analysis on (3.3):

$$\frac{\partial u}{\partial t} = -\vec{V} \cdot \nabla u - c_{pd} \bar{\theta}_v \frac{\partial \pi'}{\partial x} \tag{3.38}$$

$$\frac{\partial w}{\partial t} = -\vec{V} \cdot \nabla w - c_{pd} \bar{\theta}_v \frac{\partial \pi'}{\partial z} + g \frac{\theta_v'}{\bar{\theta}_v} \tag{3.39}$$

$$\frac{\partial \theta'}{\partial t} = -\vec{V} \cdot \nabla \theta' - w \frac{d\bar{\theta}}{dz} \tag{3.40}$$

$$\frac{\partial \pi'}{\partial t} = -\frac{\bar{c_s}^2}{\bar{\rho} c_{pd} \bar{\theta}_v^2} \left[\nabla \cdot \bar{\rho} \bar{\theta}_v \vec{V} \right] \tag{3.41}$$

This is the fully compressible model framework.